Tensor Methods in the Representation of SU(2)

By

GEORGE EDWARD IVEY

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA IN PARTIAL
FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1972

UNIVERSITY OF FLORIDA
3 1262 08552 5771

DEDICATION

This dissertation is dedicated to the author's wife, Sandra, and to his parents, Mr. and Mrs. George E. Ivey, Sr.

ACKNOWLEDGMENTS

The author wishes to thank Dr. Charles Luehr for his encouragement and great assistance. He also wishes to thank Dr. James Brooks for his inspiration as a teacher and for chairing his committee. Also deserving of thanks are Dr. Zoran Pop-Stojanovic, Dr. Kermit Sigmon, Dr. Arun Varma and Dr. Charles Hooper for serving as members of his committee. Finally, thanks go to Mrs. Jeanne Ojeda for typing this dissertation.

TABLE OF CONTENTS

		Page
ACKNOWLEDGMENTS		iii
ABSTRACT		vi
CHAPTERS:		
I	INTRODUCTION	1
	1. Lie Groups and Quantum Mechanics	1
	2. Lie Algebras	3
	3. Representations of Groups	4
II	MATRIX REPRESENTATIONS OF SU(2)	7
	1. SU(2)	7
	2. Other Representations of the Rotation Group	9
	3. Calculation of the Clebsch-Gordon Coefficients	10
III	TENSOR REPRESENTATIONS OF SU(2)	11
	1. Unitary Geometry	11
	2. Linear Transformations on U_2	12
	3. A Representation of su(2)	13
	4. Further Tensor Representations of su(2)	15
	5. Irreducibility of the Representations	16
	6. Physical Interpretation	19
IV	COMPUTATION OF THE CLEBSCH-GORDON COEFFICIENTS	21
	1. Method of Computation	21
	2. The Basis Tensors for $(U_2^{\alpha 2j})_s$	22

TABLE OF CONTENTS (CONTINUED)

Page

3. A Basis for $(U_2^{\boxtimes 2j_1})_s \boxtimes (U_2^{\boxtimes 2j_2})_s$ 24

4. Special Cases 24

5. The General $\Phi_{j_1 j_2}(J,M)$ 31

6. Allowed Values of J and M 38

7. Special Cases 38

8. Conclusion 41

BIBLIOGRAPHY 43

BIOGRAPHICAL SKETCH 44

Abstract of Dissertation Presented to the
Graduate Council of the University of Florida in Partial Fulfillment
of the Requirements for the Degree of Doctor of Philosophy

TENSOR METHODS IN THE REPRESENTATION OF SU(2)

By

George Edward Ivey

June, 1972

Chairman: James K. Brooks
Co-Chairman: Charles P. Luehr
Major Department: Mathematics

The Lie group SU(2), the two by two unitary matrices with determinant one, was originally used in quantum physics as a representation of the rotation group. Calculation of SU(2) and higher order representations of the rotation group give information about operators which are invariant under rotations. More recently, SU(2), SU(3) and other groups have been used in the theory of elementary particles.

Products of irreducible representations of the rotation group are not, in general, irreducible. They can, however, be written as a sum of irreducible representations. In this case, eigenfunctions of the angular momentum operators corresponding to the product representation can be written as a sum of eigenfunctions of the angular momentum operators corresponding to the irreducible representations with coefficients called the Clebsch-Gordon coefficients. It is the purpose of this dissertation to construct a simpler representation and to give a method of evaluating the Clebsch-Gordon coefficients more direct than the usual one.

The first two chapters are devoted to defining the rotation group and its usual matrix representation, SU(2). An isomorphism is

constructed from the rotation group to SU(2) by assigning to each rotation a matrix which affects the basis vectors in a three-dimensional vector space in the same ways as the rotation affects the coordinate axes. In addition, the Lie algebra su(2) associated with SU(2) is defined and its relationship with the rotation group is given.

In the third chapter, attention is concentrated on su(2). A representation of su(2) in terms of tensors is given. The basic representation consists of dyadics defined on U_2, the two-dimensional unitary space. Irreducible representations of higher order consist of operators on symmetric tensors from the tensor products of U_2 with itself. Products of irreducible representations are given by operators on tensors which are symmetric only under certain iterchanges of indices.

In this representation, it is possible to construct the eigenfunctions of those operators corresponding to angular momentum operators. In the fourth, they are, in fact, constructed in such a way as to form an orthonormal basis for the representation spaces. In this manner, evaluating the Clebsch-Gordon coefficients becomes simply a matter of calculating the normalizing factors for the eigenfunctions.

CHAPTER I

INTRODUCTION

1. Lie Groups and Quantum Mechanics

One of the most striking applications of "pure" mathematics to physics lies in the application of the theory of Lie groups and Lie algebras to quantum physics. Quantum mechanics differs from classical mechanics in that a particle is not described as a moving point in space. Instead the particle is described by means of a wave function. The components of momentum, energy, angular momentum, etc., are considered to be operators on the wave function. The measurable quantities associated with each operator are the eigenvalues of the operators. Thus a basic problem in quantum physics is the solution of the time-independent Schroedinger equation for the eigenvalues and eigenfunctions of H, the Hamiltonian operator which represents the sum of the kinetic and potential energy of the system.

The difficulty involved in solving this equation depends entirely upon the form of the Hamiltonian. For example, consider a problem involving four electrons orbiting about a fixed nucleus. The Hamiltonian would have terms describing the energy of the nucleus and each of the electrons, and all the interaction terms of any two particles together. If any computer were able to solve this problem

numerically, the solution of reasonable accuracy would require approximately 10^{24} entries [9] which would fill all of the libraries in the country.

It is necessary, then, in most cases, to simplify a problem before a useful solution or qualitative information can be gained. This can often be done by utilizing the symmetry properties of the Hamiltonian. To investigate these symmetries the concept of a Lie group of transformations is introduced.

A transformation is an operator which induces a change in the coordinates. The group is said to be a Lie group if its elements depend continuously on certain parameters. Every rotation about a given axis depends continuously on the angles defining the axis and the angle giving the amount of rotation. Thus, the group of all rotations is a Lie group of transformations.

The problem with which this paper is primarily concerned is that of one or two electrons in a spherically symmetrical potential in which the interactions between the two electrons are neglected. Since the kinetic energy operator for each electron is proportional to the Laplacian of its coordinates which is invariant under rotations, the Hamiltonian is invariant under rotations. If R is a rotation about any axis and ψ is any wave function, then $H(R\psi) = R(H\psi)$, or, in terms of the operators alone, $HR = RH$. That is, the operators commute. Since the operators commute, it is possible to find a complete set of simultaneous eigenfunctions of both R and H [10]. That is, since rotation about any axis commutes with the Hamiltonian, one may find eigenfunctions of the Hamiltonian which are symmetric with respect

to that axis. Note that although the potential is spherically symmetric, one may not assert that the eigenfunctions are spherically symmetric. This is because, while rotations about two different axes commute with H, such rotations do not commute with one another. In this case more information may be gained by looking at the Lie Algebra.

2. Lie Algebras

Let G be a Lie group and p a representation of G by automorphisms of a finite dimensional vector space. Then if g(t) is an element of G depending on the parameter t and v is an element of the representation space, one may define an "infinitesimal generator" of G by

$$\alpha(v) = \frac{d}{dt}\, p(g(t))(v)\Big|_{t=0} .$$

The algebra of all infinitesimal generators of G is called the Lie algebra, g, associated with the Lie group G [5]. In the case of the Lie group of rotations, one may think of the elements of the Lie algebra as generating the infinitesimally small rotations. An element of the group may be generated by applying an infinitesimally small rotation an infinite number of times. The elements of the Lie algebra may also be interpreted as the angular momentum operators [9]. The x component of angular momentum, J_x, is defined in such a way that its conservation follows from the invariance of the Hamiltonian under rotations about the x axis. Then J_x may be defined in terms of the derivative with respect to rotation about the x axis:

$$J_x\psi = \left(\lim_{\delta\theta\to 0} \frac{ih}{2\pi} \frac{P_{\delta\theta}-1}{\delta\theta}\right)\psi$$

or $$P_{\delta\theta} = 1 - \frac{i2\pi}{h}(\delta\theta)J_x$$

where $P_{\delta\theta}$ is the rotation about the x axis of angle $\delta\theta$.

The operator J_x generates a finite rotation in the following way: Let $\theta = n(\delta\theta)$ and let n go to infinity while $\delta\theta$ goes to zero in such a way as to keep θ constant. P_θ is the result of n iterations of $P_{\delta\theta}$, so

$$\begin{aligned} P_\theta\psi &= \lim (P_{\delta\theta})^n\psi \\ &= \lim \left[1 - \frac{2\pi i\delta\theta}{h} J_x\right]^n\psi \\ &= \lim \left[1 - \frac{2\pi i\theta}{hn} J_x\right]\psi \\ &= \sum_j \left(\frac{-2\pi i\theta}{h} J_x\right)^j \frac{1}{j!}\psi \\ &= e^{-\frac{2\pi i\theta}{h} J_x}\psi \ . \end{aligned}$$

Similar calculations give the same result for rotations about the y and z axes. The Lie algebra of the rotation group is a three-dimensional algebra with basis elements of J_x, J_y, and J_z.

3. Representations of Groups

A representation of a group G is a homomorphism from the group G to a group of operators on a vector space. The representation is said to be faithful if the homomorphism is an isomorphism [1]. A

representation of a Lie group leads to an equivalent representation of the associated Lie algebra. The most common representations of the rotation group are in terms of matrices. In this paper the representations will be in terms of operators on spaces of tensors.

A representation is said to be reducible if the associated vector space contains an invariant subspace which is not the whole space or the trivial subspace. That is, all of the operators in the representation map every vector in the subspace back into the subspace. A representation is said to be irreducible if there exists no invariant subspace and completely reducible if the vector space can be written as the direct sum of all the invariant subspaces.

Of great importance in quantum physics is the question of combining two representations of the angular momentum operators (addition of angular momenta). Suppose one is interested in the total angular momentum of two electrons in a spherically symmetric potential in which the interaction between the two electrons is neglected. This problem may be simplified by considering the representation of the rotation group on each electron separately. They may be combined by taking the Kroneker product [1]. This product is not irreducible but is completely reducible. Let R in G be represented by $D_k(R)$ and $D_j(R)$ under the representations Γ_k and Γ_j respectively. $D_k(R)$ is an operator on the vector space $V_k(R)$ and $D_j(R)$ is an operator on the vector space $V_j(R)$. $D_{k,j}(R)$, the representative of R under $\Gamma_k \times \Gamma_j$, the Kroneker product of Γ_k and Γ_j, is an operator on the vector space $V_k \otimes V_j$ into the direct

sum α V_ℓ of invariant subspaces and $D_{k,j}(R) = \sum D_\ell(k)$ where D_ℓ of R is the representative of R on V_ℓ.

Of particular importance is the decomposition of the eigenfunctions of each operator in a product representation. An eigenfunction in $V_j \alpha V_k$ may be written as a sum, with certain coefficients, of eigentensors in the invariant spaces V_ℓ. These coefficients are called the Clebsch-Gordon coefficients. It will be the purpose of this paper to give a straightforward method of calculating the Clebsch-Gordon coefficients.

CHAPTER II

MATRIX REPRESENTATIONS OF SU(2)

1. SU(2)

The adjoint of a matrix is the matrix derived by exchanging the indices of each component and taking the complex conjugate. Thus, if a matrix has components u_{ij}, its adjoint has components $\bar{u}_{ji}$. A matrix is said to be unitary if its adjoint is equal to its inverse. SU(2) may then be defined as the group of unitary two by two matrices with determinant one. SU(2) gives a representation of the rotation group in the following way: Let $P(\alpha,\beta,\gamma)$ be the rotation with Eulerian angles α, β, γ. This is the rotation achieved by rotating through an angle γ around the z axis, through an angle β around the y axis and then through an angle α around the z axis again [10]. The matrix associated with this rotation is

$$u = \begin{bmatrix} e^{i(\alpha+\gamma)/2} \cos \beta/2 & e^{i(\gamma-\alpha)/2} \sin \beta/2 \\ -e^{-i(\gamma-\alpha)/2} \sin \beta/2 & e^{-i(\alpha+\gamma)/2} \cos \beta/2 \end{bmatrix}.$$

This particular choice of the Eulerian angles has the property of diagonalizing rotations about the z axis. That is, if $\beta = 0$

$$u = \begin{bmatrix} e^{\frac{i(\alpha+\gamma)}{2}} & 0 \\ 0 & e^{-\frac{i(\alpha+\gamma)}{2}} \end{bmatrix}.$$

It should also be noted that this representation is double-valued. The Eulerian angles are determined only to within 2π so the half angles are determined only to within π. Since the trigonometric functions change sign when the argument is changed by π, both the matrices u and -u represent the same rotation.

One may now derive the corresponding representation of the Lie algebra associated with the rotation group as su(2), the Lie algebra associated with SU(2). If the angular momentum operator about the z axis, J_z, is associated, as before, with the infinitesimal generator of rotations about the z axis, and P_θ is arbitrary rotation about the z axis

$$P_\theta = e^{-\frac{2\pi i\theta}{h} J_z}$$

so if σ_z is the matrix representing J_x, one has

$$U(\theta) = e^{-\frac{2\pi i\theta}{h} \sigma_z} .$$

Since $U(\theta)$ must be a two by two unitary matrix with determinant one, σ_x must be a two by two matrix with zero trace [5]. A process of diagonalization shows that this is true of any element of the Lie algebra. Therefore, su(2) is the three-dimensional algebra of two by two matrices with zero trace. The standard choice of basis is the Pauli matrices:

$$\sigma_x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} , \quad \sigma_y = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} , \quad \sigma_z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} .$$

2. Other Representations of the Rotation Group

The two by two matrices may be considered as operators on a two-dimensional complex vector space. If x_1 and x_2 are two basis vectors, then

$$x_i' = U\, x_i = \sum_j x_j\, u_{ji} .$$

If U is the matrix

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

then $x_1' = ax_1 - \bar{b}x_2$, $\quad x_2' = bx_1 + \bar{a}x_2$.

Representations of SU(2) of higher dimensionality may be generated by considering matrices on a vector space whose basis vectors are the n+1 monomials of degree n in x_1 and x_2, i.e., $f_{np} = x_1^p\, x_2^{n-p}$. Then the matrix U_n corresponding to U in the original group is defined by

$$\begin{aligned} U_n(f_{np}(x_1,x_2)) &= f_{np}(x_1',x_2') \\ &= (ax_1 - \bar{b}x_2)^p\ (bx_1 + \bar{a}x_2)^{n-p} \\ &= f_{np}(U(x_1,x_2)) . \end{aligned}$$

These matrices, after a normalization to make them unitary, as n takes on all integral values, include all the distinct, irreducible, unitary representations of SU(2) [10]. Since SU(2) is a faithful representation of the rotation group, this gives a complete set of irreducible representations of the rotation group.

3. Calculation of the Clebsch-Gordon Coefficients

As stated above when two irreducible representations of a group are combined in the Kroneker product, the result is not necessarily irreducible. It is however completely reducible. That is, it can be written as the sum of completely irreducible representations. Eigenfunctions in the combined representation may be written as a sum of eigenfunctions in the irreducible representation with the Clebsch-Gordon coefficients.

Standard methods of finding the Clebsch-Gordon coefficients involve using the properties of the rotation group itself rather than any representation, particularly the fact that the rotations can be written in terms of exponentials [11]. Calculation of the Clebsch-Gordon coefficients may be done by noting that, in the matrix representation, the product of matrices from the two individual representations is the same as that of the matrix from the combined representation multiplied by a matrix whose elements are the Clebsch-Gordon coefficients. Then a number of matrix multiplications are performed which, together with an integration over the Euler angles in the matrix representation and some algebraic manipulations, produce the Clebsch-Gordon coefficients.

In this paper, a tensor representation will be discussed which leads to a simpler and more straightforward method of calculating the Clebsch-Gordon coefficients. It also has the advantage that the possible values of the total angular momentum follow directly from the calculation and that it may be more easily generalized.

CHAPTER III

TENSOR REPRESENTATIONS OF SU(2)

1. Unitary Geometry

The unitary space U_2 is a two-dimensional vector space over the field of complex numbers. Associated with U_2 is the conjugate space $\bar{U}_2$, also two-dimensional over the complex numbers. These spaces are related by the complex conjugation operator, an anti-linear function from U_2 to $\bar{U}_2$ and from $\bar{U}_2$ to U_2. The complex conjugation of a vector u in either U_2 or $\bar{U}_2$ will be represented by $\bar{u}$. Complex conjugation satisfies:

(1) $\bar{\bar{u}} = u$

(2) $\overline{(u+v)} = \bar{u} + \bar{v}$

(3) $\overline{(au)} = \bar{a}\,\bar{u}$.

where a is a complex number and u on v are vectors in either U_2 or $\bar{U}_2$. To simplify notation, all vectors in $\bar{U}_2$ will be denoted as the complex conjugate, $\bar{u}$, of some vector u in U_2.

Also, there is defined a scalar product $\bar{u}\cdot v$ with $\bar{u} \in \bar{U}_2$, $v \in U_2$ with the properties:

(1) $\bar{u}\cdot v = v\cdot\bar{u}$

(2) $(a\bar{u})\cdot v = a(\bar{u}\cdot v)$

(3) $(\bar{u}+\bar{v})\cdot w = \bar{u}\cdot w + \bar{v}\cdot w$

(4) $\bar{u}\cdot(av) = a(\bar{u}\cdot v)$

(5) $\bar{u}\cdot(v+w) = \bar{u}\cdot v + \bar{u}\cdot w$

(6) $\overline{(\bar{u}\cdot v)} = u\cdot\bar{v}$

(7) $\bar{u}\cdot u \geq 0$

(8) $\bar{u}\cdot u = 0$ if and only if $u = 0$.

2. Linear Transformations on U_2

Any linear transformation on U_2 is identified by its action on the two basis vectors of U_2 and may be represented as a dyadic, an element of the tensor product space $U_2 \otimes \bar{U}_2$. Let g_1 and g_2 be basis vectors of U_2 such that $\bar{g}_1\cdot g_2 = \bar{g}_2\cdot g_2 = 1$ and $\bar{g}_1\cdot g_2 = \bar{g}_2\cdot g_1 = 0$. Let T be a linear transformation on U_2 defined by $T(g_1) = ag_1 + bg_2$, $T(g_2) = cg_1 + dg_2$, where a, b, c, and d are complex numbers. Then one may write

$$T = ag_1\bar{g}_1 + bg_2\bar{g}_1 + cg_1\bar{g}_2 + dg_2\bar{g}_2 .$$

Then, applying the scalar product,

$$\begin{aligned} T\cdot g_1 &= ag_1(\bar{g}_1\cdot g_1) + bg_2(\bar{g}_1\cdot g_1) + cg_1(\bar{g}_2\cdot g_1) + dg_2(\bar{g}_2\cdot g_1) \\ &= ag_1 + bg_2 \\ T\cdot g_2 &= ag_1(\bar{g}_1\cdot g_2) + bg_2(\bar{g}_1\cdot g_1) + cg_1(\bar{g}_2\cdot g_2) + dg_2(\bar{g}_2\cdot g_2) \\ &= cg_1 + dg_2 . \end{aligned}$$

The tensors $g_1\bar{g}_1$, $g_2\bar{g}_1$, $g_1\bar{g}_2$, and $g_2\bar{g}_2$ are basis elements of the tensor space $U_2 \otimes \bar{U}_2$.

Many of the properties of matrices may be extended to dyadics. The [illegible] adjoint of a dyadic is defined as the dyadic produced

by taking the complex conjugate of each term and reversing its order. For example, the Hermitian adjoint of $g_1\bar{g}_2$ is $g_2\bar{g}_1$. A dyadic which is identical with its own adjoint is called Hermitian. The trace of a dyadic is formed by taking the scalar product in place of the tensor product. For example, the trace of $g_1\bar{g}_1$ is $g_1\cdot\bar{g}_1 = 1$, the trace of $g_1\bar{g}_2$ is $g_1\cdot\bar{g}_2 = 0$.

3. A Representation of su(2)

Because of the close relationship between a Lie group and its associated Lie algebra, every representation of the group corresponds to a representation of the associated Lie algebra. Since, in this case, it is simpler to find a representation of su(2), the Lie algebra associated with SU(2), this is the method that will be used.

In the previous chapter, the Pauli matrices σ_x, σ_y, and σ_z were given as a basis for su(2). Since these are 2x2 matrices, they also define transformations on U_2. That is, σ_x gives the transformation defined by $\sigma_x g_1 = g_2$, $\sigma_x g_2 = g_1$; σ_y gives the transformation such that $\sigma_y g_1 = ig_2$, $\sigma_y g_2 = -ig_1$; σ_z gives the transformation such that $\sigma_z g_1 = g_1$, $\sigma_z g_2 = -g_2$.

In order to keep the eigenvalues the same as those used in quantum physics, the basis elements for this representation will be those given by σ_x, σ_y, and σ_z multiplied by $\frac{1}{2}$. That is:

$$J_x = \frac{1}{2}(g_2\bar{g}_1 + g_1\bar{g}_2)$$

$$J_y = \frac{i}{2}(g_2\bar{g}_1 - g_1\bar{g}_2)$$

$$J_z = \frac{1}{2}(g_1\bar{g}_1 - g_2\bar{g}_2) .$$

Since the matrix representation of su(2) is irreducible and this association of σ_x with J_x, etc., defines an isomorphism from the space of 2x2, Hermitian, traceless matrices to the subspace of Hermitian, traceless dyadics of $U_2 \alpha \bar{U}_2$, this latter space forms an irreducible representation of su(2).

With this particular choice of basis, the eigenvectors of J_z are just those of the form ag_1 and bg_2 with a and b complex numbers:

$$J_z(ag_1) = \frac{1}{2}\, a(g_1\bar{g}_1 \cdot g_1 - g_2\bar{g}_2 \cdot g_1) = \frac{1}{2}\, a\bar{g}_1$$

$$J_z(bg_2) = \frac{1}{2}\, a(g_1\bar{g}_1 \cdot g_2 - g_2\bar{g}_2 \cdot g\ ) = -\frac{1}{2}\, bg_2$$

with eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$ respectively.

Also of interest in the physical interpretation is the dyadic J^2 defined by

$$J^2 = (J_xJ_x + J_yJ_y + J_zJ_z)\ .$$

J^2 has the properties:

$$\begin{aligned} J^2g_1 &= J_xJ_xg_1 + J_yJ_yg_1 + J_zJ_zg_1 \\ &= \frac{1}{2}\, J_xg_2 + \frac{i}{2}\, J_yg_2 + \frac{1}{2}\, J_zg_1 \\ &= \frac{1}{4}\, g_1 + \frac{1}{4}\, g_1 + \frac{1}{4}\, g_1 \\ &= \frac{3}{4}\, g_1 \end{aligned}$$

$$\begin{aligned} J^2g_2 &= J_xJ_xg_2 + J_yJ_yg_2 + J_zJ_zg_2 \\ &= +\frac{1}{2}\, J_xg_1 - \frac{i}{2}\, J_yg_1 - \frac{1}{2}\, J_zg_2 \\ &= \frac{1}{4}\, g_2 + \frac{1}{4}\, g_2 + \frac{1}{4}\, g_2 \\ &= \frac{3}{4}\, g_2\ . \end{aligned}$$

Since the two basis vectors g_1 and g_2 are both eigenvectors of J^2, with the same eigenvalue, $\frac{3}{4}$, all vectors in U_2 are eigenvectors of J^2 with eigenvalue $\frac{3}{4}$. (It should be noted, however, that J^2 is not a traceless dyadic, and, therefore, not in the representation of su(2).)

4. Further Tensor Representations of su(2)

The tensor space $(U_2^{\boxtimes 2j})_s$ is the space of insert on all symmetric tensor for j=0, $\frac{1}{2}$, 1, $\frac{3}{2}$, $\cdots$. Since a basis for $(U_2^{\boxtimes 2j})$ may be formed by taking all tensor products of basis vectors from 2j copies of U_2, a basis for $(U_2^{\boxtimes 2j})_s$ may be formed by taking all symmetric combinations of such products [6]. $(U_2^{\boxtimes 2j})_s$ is a unitary space if complex conjugation is defined by taking the complex conjugate of each vector from U_2 appearing in the tensor. That is, the complex conjugate of the symmetric tensor $g_1g_2 + g_2g_1$ is $\bar{g}_1\bar{g}_2 + \bar{g}_2\bar{g}_1$. The scalar product $\bar{u}:v$ of two tensors u and v in $(U_2^{\boxtimes 2j})_s$ is defined by taking the scalar product of the first vector in each tensor product, the scalar product of the second vectors, etc., and then multiplying them all together.

To construct representations of higher dimension, consider the linear operators on the space $(U_2^{\boxtimes 2j})_s$. To the linear operator J_α, α = x, y, or z, one associates $D_j(J_\alpha)$ defined by

$$D_j(J_\alpha) = J_\alpha \boxtimes I^{\boxtimes(2j-1)} + I \boxtimes J_\alpha \boxtimes I^{\boxtimes(2j-2)} + \cdots + I^{\boxtimes(2j-1)} \boxtimes J_\alpha$$

where I is the unit dyadic $g_1\bar{g}_1 + g_2\bar{g}_2$.

Since the map of J_α into $D_j(J_\alpha)$ is a vector space homomorphism, this gives a representation of su(2) for each value of j. For j=0, this gives the trivial representation in which all elements of su(2) are mapped into 0. When $j = \frac{1}{2}$, the representation as dyadics, given above, is obtained.

Hereafter, the basis elements of $(U_2^{\alpha(2j)})_s$ will be denoted by g(a,b), where g(a,b) is the sum of all monomials in g_1 and g_2 with g_1 appearing a times and g_2 appearing b times, a + b = 2j. For example, in U_2, $g(1,0) = g_1$ and $g(0,1) = g_2$ are the basis vectors. In $(U_2^{\alpha 2})_s$, $g(2,0) = g_1g_1$, $g(1,1) = g_1g_2 + g_2g_1$, and $g(0,2) = g_2g_2$ are the basis tensors. It may be noted that the space $(U_2^{\alpha(2j)})_s$ is of dimension 2j + 1.

5. Irreducibility of the Representations

A representation is irreducible if there is no subspace, other than the entire space and the 0 space, of the representation space which is invariant under the action of all of the operators in the representation [1]. To prove the irreducibility of the tensor representations, it is sufficient to show that there is no nontrivial subspace of $(U_2^{\alpha 2j})_s$ which is invariant under $D_j(J_x)$, $D_j(J_y)$, and $D_j(J_z)$.

Since $D_j(J_z) = J_z \alpha I^{\alpha(2j-1)} + \cdots + I^{\alpha(2j-1)} \alpha J_z$ acts individually on g_1 and g_2, the effect of $D_j(J_z)$ on a basis tensor g(a,b) may be determined by examining the effect on a single monomial term of g(a,b). If $D_j(J_z)$ is applied to such a monomial term, the result is a sum of such terms. The g_1 and g_2's are left unchanged with a factor of $\frac{1}{2}$ if the J_x was acting on a g_1 and $-\frac{1}{2}$ if the J_z was acting on a g_2. Since each term contains a g_1's and b g_2's, one has

$$D_j(J_z)\, g(a,b) = \frac{a-b}{2}\, g(a,b) \ .$$

Every basis vector is an eigentensor of $D_j(J_z)$ with eigenvalue $\frac{a-b}{2}$ so every subspace spanned by a subset of the basis tensors g(a,b) is invariant under $D_j(J_z)$. Let S be a subspace of $(U_2^{\otimes 2j})_s$ invariant under $D_j(J_z)$. Since $D_j(J_z)$ is Hermitian, S*, the orthogonal complement of S, is also invariant under $D_j(J_z)$ [4]. Let g(a,b) be any basis tensor in $(U_2^{\otimes 2j})_s$. Then g(a,b) can be written uniquely as a sum of tensors in S and S*: $g(a,b) = u + v$. Applying $D_j(J_z)$ to this equation gives $\frac{a-b}{2}\, g(a,b) = D_j(J_z)u + D_j(J_z)v$. Since S and S* are invariant under $D_j(J_z)$ and the sum is unique, $D_j(J_z)u = \frac{a-b}{2}\, u$ and $D_j(J_z)v = \frac{a-b}{2}\, v$. That is, both u and v are eigentensors of $D_j(J_z)$ with eigenvalue $\frac{a-b}{2}$. Since the dimension of $(U_2^{\otimes 2j})_s$ is 2j+1 and there are already 2j+1 eigentensors with distinct eigenvalues, it is impossible for there to be two orthogonal eigentensors with the same eigenvalue. Thus, either u or v is zero and every basis tensor is contained in either S or S*. If S has dimension n, S* has dimension 2j+1-n so S* can contain at most 2j+1-n of the 2j+1 tensors g(a,b). The S must contain exactly n of these tensors. Therefore, every subspace of $(U_2^{\otimes 2j})_s$, invariant under $D_j(J_z)$, has a basis which consists of eigenvectors of $D_j(J_z)$.

Now attention is turned to $D_j(J_x)$ and $D_j(J_y)$. By the same reasoning as with $D_j(J_z)$, it is sufficient to calculate the effect of $D_j(J_x)$ and $D_j(J_y)$ on a single monomial of the basis tensor g(a,b). $D_j(J_x)$ acting on such a term again results in a sum of such terms;

however, now in addition to the factor of $\frac{1}{2}$, each g_1 is changed to a g_2 and vice versa. Each term from g(a,b) is changed into terms from g(a+1,b-1) and g(a-1,b+1). Since g(a,b) has $\frac{(a+b)!}{a!b!}$ monomial terms while g(a+1,b-1) has $\frac{(a+b)!}{(a+1)!(b-1)!}$ terms and g(a-1,b+1) has $\frac{(a+b)!}{(a-1)!(b+1)!}$, collecting all the terms properly gives

$$D_j(J_x)\ g(a,b) = \frac{a}{2}\left\{\frac{(a+b)!}{a!b!}\right\}\left\{\frac{(a-1)!(b+1)!}{(a+b)!}\right\} g(a-1,b+1)$$

$$+\frac{b}{2}\left\{\frac{(a+b)!}{a!b!}\right\}\left\{\frac{(a+1)!(b-1)!}{(a+b)!}\right\} g(a+1,b-1)$$

$$= \frac{b+1}{2}\, g(a-1,b+1) + \frac{a+1}{2}\, g(a+1,b-1)\,.$$

By essentially the same procedure,

$$D_j(J_y)\ g(a,b) = \frac{(b+1)i}{2}\, g(a-1,b+1) - \frac{(a+1)i}{2}\, g(a+1,b-1)\,.$$

Finally, let S be a nonzero subspace of $(U_2^{\otimes 2j})_s$ which is invariant under $D_j(J_x)$, $D_j(J_y)$, and $D_j(J_z)$. Since it is invariant under $D_j(J_z)$, there exists a basis of S consisting of the g(a,b) tensors. Since S is invariant under $D_j(J_y)$ and $D_j(J_x)$, it is also invariant under the "step up" and "step down" operators: $D_j(J_+) = D_j(J_x) + iD_j(J_y)$, $D_j(J_-) = D_j(J_x) - D_j(iJ_y)$. Let g(a,b) be a basis vector for S and g(a',b') any basis vector of $(U_2^{\otimes(2j)})_s$. If a' = a, since a+b = a' + b' = 2j, b = b' so g(a',b') is in S. If a' is larger than a, apply $D_j(J_+)$ to g(a,b) a' - a times.

Since

$$D_j(J_+)\, g(a,b) = J_x\, g(a,b) + iJ_y\, g(a,b)$$

$$= \frac{b+1}{2}\, g(a-1,b+1) + \frac{a+1}{2}\, g(a+1,b-1) - \frac{b+1}{2}\, g(a-1,b+1)$$

$$+ \frac{a+1}{2}\, g(a+1,b-1)$$

$$= (a+1)\, g(a+1,b-1)$$

so $\{D_j(J_+)\}^{a'-a}\, g(a,b) = (a+1)(a+2)\cdots(a')\, g(a',b')$.

Since S is invariant under $D_j(J_+)$, $g(a',b')$ is in S.

If a' is less than a, apply J_- a-a' times.

Since

$$D_j(J_-)\, g(a,b) = D_j(J_x)\, g(a,b) - iD_j(J_y)\, g(a,b)$$

$$= \frac{a+1}{2}\, g(a+1,b-1) + \frac{b+1}{2}\, g(a-1,b+1) + \frac{b+1}{2}\, g(a-1,b+1)$$

$$- \frac{a+1}{2}\, g(a+1,b-1)$$

$$= b+1\; g(a-1,b+1)$$

so $\{D_j(J_-)\}^{a-a'}\, g(a,b) = (b+1)(b+2)\cdots(b')\, g(a',b')$.

Therefore, in all cases $g(a',b')$ is in S. Since $g(a',b')$ was any basis tensor in $(U_2^{\otimes 2j})_s$, $S = (U_2^{\otimes 2j})_s$ and the representation is irreducible.

6. Physical Interpretation

The tensor representations given may be interpreted in terms of quantum physics. The operators $D_j(J_x)$, $D_j(J_y)$, $D_j(J_z)$ are interpreted as the angular momentum operators about the x, y, and z axes

respectively. $D_j(J^2) = (D_j(J_x))^2 + (D_j(J_y))^2 + (D_j(J_z))^2$ is the total angular momentum operator. The tensor space $(U_2^{\alpha 2j})_s$ is the space of wave functions with total angular momentum j. As will be seen later, every tensor in $(U_2^{\alpha 2j})_s$ is an eigentensor of $D_j(J^2)$ with eigenvalue $j(j+1)$.

The eigenvalues of $D_j(J_z)$ are the possible values of the z component of angular momentum. In keeping with notational convention, eigenvalues of $D_j(J_z)$ will hereafter be denoted by the letter m. Thus the basic vectors g(a,b) represent wave functions with definite values of the z component of angular momentum. $D_j(J_z)\ g(a,b) = mg(a,b)$ where $m = \frac{a-b}{2}$. Since the only restrictions on a and b are that they be positive integers and $a+b = 2j$, the possible values of m may be obtained by letting b range from 0 to 2j. Therefore, m must be $\frac{2j-0}{2} = j$, $\frac{(2j-1)-1}{2} = j-1$, $\frac{(2j-2)-2}{2} = j-2$, $\cdots$, or $\frac{0-2j}{2} = -j$. That is, m may have values at integer steps between $-j$ and $+j$.

CHAPTER IV

COMPUTATION OF THE CLEBSCH-GORDON COEFFICIENTS

1. Method of Computation

The Kroneker product of two representations D_{j_1} and D_{j_2} gives the representation in which $D_{j_1 j_2}(J_\alpha) = D_{j_1}(J_\alpha) \times I^{\alpha 2 j_2} + I^{\alpha 2 j_1} \times D_{j_2}(J_\alpha)$, where α = x, y, or z, is an operator over the tensor space $(U_2^{\alpha 2 j_1})_s \times (U_2^{\alpha 2 j_2})_s$. This representation is not irreducible but may be written as the sum of irreducible representations, the Clebsch-Gordon series.

This means the tensor space $(U_2^{\alpha 2 j_1})_s \times (U_2^{\alpha 2 j_2})_s$ may be written as the direct sum of irreducible spaces and the eigenvectors of $D_{j_1 j_2}(J_z)$ may be written as a sum of eigenvectors of $D_j(J_z)$ for certain values of j. The Clebsch-Gordon coefficients are the coefficients involved in this last expansion.

Specifically, let $\phi(j_1,m_1)$ and $\phi(j_2,m_2)$ be tensors from orthonormal bases of $(U_2^{\alpha 2 j_1})_s$ and $(U_2^{\alpha 2 j_2})_s$, respectively, which are simultaneously eigentensors of $D_j(J_z)$ and $D_j(J^2)$. Then products of the type $\phi(j_1,m_1) \times \phi(j_2,m_2)$ form an orthonormal basis for $(U_2^{\alpha 2 j_1})_s \times (U_2^{\alpha 2 j_2})_s$. Such basis tensors will not in general be simultaneous eigentensors of $D_{j_1 j_2}(J_z)$ and $D_{j_1 j_2}(J^2) = (D_{j_1 j_2}(J_x))^2 + (D_{j_1 j_2}(J_y))^2 + (D_{j_1 j_2}(J_z))^2$.

Let $\phi_{j_1 j_2}(J,M)$ form a basis for the space $(U_2^{\alpha 2 j_1})_s \times (U_2^{\alpha 2 j_2})_s$ which are simultaneous eigentensors of $D_{j_1 j_2}(J_z)$ and $D_{j_1 j_2}(J^2)$. Since

tensors of the form $\phi(j_1,m_1) \otimes \phi(j_2,m_2)$ form an orthonormal basis for $(U_2^{\otimes 2j_1})_s \otimes (U_2^{\otimes 2j_2})_s$, one has

$$\Phi_{j_1 j_2}(J,M) = \sum_{m_1,m_2} C_{j_1 j_2}(m_1,m_2;J,M)\ \phi(j_1,m_1) \otimes \phi(j_2,m_2)$$

where the coefficients $C_{j_1 j_2}(m_1,m_2;J,M)$ are just the Clebsch-Gordon Coefficients. Finally, since all the bases involved are orthonormal,

$$C_{j_1 j_2}(m_1,m_2;J,M) = (\bar{\phi}(j_1,m_1) \otimes \bar{\phi}(j_2,m_2)) : \Phi_{j_1 j_2}(J,M) .$$

2. The Basis Tensors for $(U_2^{\otimes 2j})_s$

It has been seen that the tensors $g(a,b)$ form a basis for $(U_2^{\otimes 2j})_s$ and are eigentensors of $D_j(J_z)$ with eigenvalue $\frac{a-b}{2}$. It remains only to show that they are also eigentensors of $D_j(J^2)\cdot D_j(J^2) = (D_j(J_x))^2 + (D_j(J_y))^2 + (D_j(J_z))^2$ and the action of $D_j(J_x)$, $D_j(J_y)$, and $D_j(J_z)$ have already been calculated.

$$\begin{aligned} D_j(J^2)\ g(a,b) &= (D_j(J_x))^2\ g(a,b) + (D_j(J_y))^2\ g(a,b) + (D_j(J_z))^2\ g(a,b) \\ &= D_j(J_x)\ \left(\frac{a+1}{2}\ g(a+1,b-1) + \frac{b+1}{2}\ g(a-1,b+1)\right) \\ &\quad + D_j(J_y)\left(\frac{a+1}{2}\ ig(a+1,b-1) + \frac{b+1}{2}\ ig(a-1,b+1)\right) \\ &\quad + D_j(J_z)\ \left(\frac{a-b}{2}\ g(a,b)\right. \\ &= \frac{(a+1)(a+2)}{4}\ g(a+2,b-2) + \frac{(a+1)b}{4}\ g(a,b) \\ &\quad + \frac{(b+1)(b+2)}{4}\ g(a-2,b+1) + \frac{a(b+1)}{4}\ g(a,b) \end{aligned}$$

$$+\frac{(a+1)b}{4}\,g(a,b)-\frac{(a+1)(a+2)}{4}\,g(a+2,b-2)+\frac{a(b+1)}{4}\,g(a,b)$$

$$-\frac{(b+1)(b+2)}{4}\,g(a-2,b+2)+\frac{(a-b)^2}{4}\,g(a,b)$$

$$=\left\{\frac{(a+1)b+a(b+1)}{2}+\frac{(a-b)^2}{4}\right\}g(a,b)$$

$$=\frac{a^2+2ab+b^2+2a+2b}{2}\,g(a,b)$$

$$=\left\{\frac{a+b}{2}\right\}\left\{\frac{a+b+2}{2}\right\}\,g(a,b)\,.$$

Since $a+b=2j$, every tensor $g(a,b)$ is an eigentensor of $D_j(J^2)$ with eigenvalue $j(j+1)$.

Since the tensors $g(a,b)$ are all eigentensors of $D_j(J_z)$ with distinct eigenvalues, they are all orthogonal. To normalize them, note that $\bar{g}_1\cdot g_2=\bar{g}_2\cdot g_1=0$ and $\bar{g}_1\cdot g_1=\bar{g}_2\cdot g_2=1$. The only nonzero terms in $\overline{g(a,b)}\cdot g(a,b)$ will be from the scalar product of those monomial with the g_1's and g_2's in exactly the same order. Each such scalar product is one, so $\overline{g(a,b)}\cdot g(a,b)$ equals the total number of monomial terms in $g(a,b)$. Since $g(a,b)$ contains all possible combinations of a g_1's and b g_2's, this is just $\frac{(a+b)!}{a!b!}=\frac{(2j)!}{a'b'}$. Therefore $\overline{g(a,b)}\cdot g(a,b)=\frac{(2j)!}{a!b!}$.

Now let $\phi(j,m)=\left\{\frac{a!b!}{(2j)!}\right\}^{1/2}g(a,b)$ where $a+b=2j$, $a-b=2m$. The $\phi(j,m)$, with m taking on values at integer steps between $-j$ and j, form an orthonormal basis for $(U_2^{\otimes 2j})_s$. They are eigentensors for both $D_j(J_z)$ and $D_j(J^2)$ with $D_j(J_z)\,\phi(j,m)=m\phi(j,m)$ and $D_j(J^2)\,\phi(j,m)=j(j+1)\,\phi(j,m)$.

3. A Basis for $(U_2^{\alpha 2j_1})_s \alpha (U_2^{\alpha 2j_2})_s$

Let $\phi(j_1,m_1)$ with m_1 running from $-j_1$ to j_1 and $\phi(j_2,m_2)$ with m_2 running from $-j_2$ to j_2 be the bases described above for $(U_2^{\alpha 2j_1})_s$ and $(U_2^{\alpha 2j_2})_s$. Then all tensor products of the form $\phi(j_1,m_1) \alpha \phi(j_2,m_2)$ for all permissible values of m_1 and m_2 form a basis for $(U_2^{\alpha 2j_1})_s \alpha (U_2^{\alpha 2j_2})$. This basis is already orthonormal since $(\phi(j_1,m_1) \alpha \phi(j_2,m_2))$: $(\phi(j_1',m_1') \alpha \phi(j_2',m_2')) = (\bar{\phi}(j_1,m_1):\phi(j_1',m_1'))(\bar{\phi}(j_2,m_2):\phi(j_2',m_2'))$ and the $\phi(j,m)$ tensors are orthonormal.

These tensors are not in general eigentensors of $D_{j_1j_2}(J^2)$. For example, the first term $(D_{j_1j_2}(J_x))^2$ expands as

$$(D_{j_1j_2}(J_x))^2 = (D_{j_1}(J_x))^2 \alpha I^{\alpha 2j_2} + I^{\alpha 2j_1} \alpha (D_{j_2}(J_x))^2 + 2D_{j_1}(J_x) \alpha D_{j_2}(J_x) .$$

The first two terms will combine with the corresponding terms due to $(D_{j_1j}(J_y))^2$ and $(D_{j_1j_2}(J_z))^2$ to give $D_{j_1}(J^2) \alpha I^{\alpha 2j_2} + I^{\alpha 2j_1} \alpha D_{j_2}(J^2)$ for which the $\phi(j_1,m_1) \alpha \phi(j_2,m_2)$ are eigentensors but the cross terms $2D_{j_1}(J_x) \alpha D_{j_2}(J_x)$ give rise to terms that do not cancel out.

4. Special Cases

Clearly for $j_1 = 0$ and $j_2 = \frac{1}{2}$ or $j_1 = \frac{1}{2}$ and $j_2 = 0$, $\phi_{\frac{1}{2}0}(J,M) = \phi_{0\frac{1}{2}}(J,M) = \phi(J,M)$, since

$$(U_2^{\alpha 1})_s \alpha (U_2^{\alpha 0})_s = (U_2^{\alpha 0})_s \alpha (U_2^{\alpha 1})_s = U_2 .$$

A more interesting special case is that in which $j_1 = \frac{1}{2}$, $j_2 = \frac{1}{2}$.

In that case, $(U_2^{\alpha 1})_s \alpha (U_2^{\alpha 1})_s = U_2 \alpha U_2$. This is simply the space of all second order tensors over U_2.

An orthonormal basis for $U_2 \alpha U_2$ consists of g_1g_1, g_1g_2, g_2g_1, and g_2g_2. In addition, one has

$$D_{\substack{11\\22}}(J_x)g_1g_2 = (J_xg_1)g_1 + g_1(J_xg_1) = \frac{1}{2}\, g_2g_1 + \frac{1}{2}\, g_1g_2$$

$$D_{\substack{11\\22}}(J_x)g_1g_2 = (J_xg_1)g_2 + g_1(J_xg_2) = \frac{1}{2}\, g_2g_2 + \frac{1}{2}\, g_1g_1$$

$$D_{\substack{11\\22}}(J_x)g_2g_1 = (J_xg_2)g_1 + g_2(J_xg_1) = \frac{1}{2}\, g_1g_1 + \frac{1}{2}\, g_2g_2$$

$$D_{\substack{11\\22}}(J_x)g_2g_2 = (J_xg_2)g_2 + g_2(J_xg_2) = \frac{1}{2}\, g_1g_2 + \frac{1}{2}\, g_2g_1 \,.$$

Similarly,

$$D_{\substack{11\\22}}(J_y)g_1g_1 = (J_yg_1)g_1 + g_1(J_yg_1) = \frac{i}{2}\, g_2g_1 + \frac{i}{2}\, g_1g_2$$

$$D_{\substack{11\\22}}(J_y)g_1g_2 = (J_yg_1)g_2 + g_1(J_yg_2) = \frac{i}{2}\, g_2g_2 - \frac{i}{2}\, g_1g_1$$

$$D_{\substack{11\\22}}(J_y)g_2g_1 = (J_yg_2)g_i + g_2(J_yg_1) = -\frac{i}{2}\, g_1g_1 + \frac{i}{2}\, g_2g_2$$

$$D_{\substack{11\\22}}(J_y)g_2g_2 = (J_yg_2)g_2 + g_2(J_yg_2) = -\frac{i}{2}\, g_1g_2 - \frac{i}{2}\, g_2g_1 \,.$$

Finally,

$$D_{\substack{11\\22}}(J_z)g_1g_1 = (J_zg_1)g_1 + g_1(J_zg_1) = \frac{1}{2}\, g_1g_1 + \frac{1}{2}\, g_1g_1 = g_1g_1$$

$$D_{\substack{11\\22}}(J_z)g_1g_2 = (J_zg_1)g_2 + g_1(J_zg_2) = \tfrac{1}{2}\, g_1g_2 - \tfrac{1}{2}\, g_1g_2 = 0$$

$$D_j(J_z)g_2g_1 = (J_zg_2)g_1 + g_2(J_zg_1) = -\tfrac{1}{2}\, g_2g_1 + \tfrac{1}{2}\, g_2g_1 = 0$$

$$D_j(J_z)g_2g_2 = (J_zg_2)g_2 + g_2(J_zg_2) = -\tfrac{1}{2}\, g_2g_2 - \tfrac{1}{2}\, g_2g_2 = -\, g_1g_2 \ .$$

Thus, all basis tensors are eigentensors of $D_{\substack{11\\22}}(J_z)$ with eigenvalues -1, 0, or +1.

To find the eigentensor of $D_{\substack{11\\22}}(J^2)$, apply it to each of the basis tensors:

$$D_{\substack{11\\22}}(J^2)g_1g_1 = (D_{\substack{11\\22}}(J_x))^2g_1g_1 + (D_{\substack{11\\22}}(J_y))^2g_1g_1 + (D_{\substack{11\\22}}(J_z))^2g_1g_1$$

$$= D_{\substack{11\\22}}(J_x)(\tfrac{1}{2}\, g_2g_1 + \tfrac{1}{2}\, g_1g_2) + D_{\substack{11\\22}}(J_y)(\tfrac{i}{2}\, g_2g_1 + \tfrac{i}{2}\, g_1g_2)$$

$$+ D_{\substack{11\\22}}(J_z)g_1g_1$$

$$= \tfrac{1}{4}\, g_1g_1 + \tfrac{1}{4}\, g_2g_2 + \tfrac{1}{4}\, g_2g_2 + \tfrac{1}{4}\, g_1g_1 + \tfrac{1}{4}\, g_1g_1$$

$$-\tfrac{1}{4}\, g_2g_2 - \tfrac{1}{4}\, g_2g_2 + \tfrac{1}{4}\, g_1g_1 + \tfrac{1}{4}\, g_1g_1$$

$$= 2g_1g_1 \ .$$

$$D_{\frac{1}{2}\frac{1}{2}}(J^2)g_1g_2 = (D_{\frac{1}{2}\frac{1}{2}}(J_x))^2 g_1g_2 + (D_{\frac{1}{2}\frac{1}{2}}(J_y))^2 g_1g_2 + (D_{\frac{1}{2}\frac{1}{2}}(J_z))^2 g_1g_2$$

$$= D_{\frac{1}{2}\frac{1}{2}}(J_x)(\tfrac{1}{2} g_2g_2 + \tfrac{1}{2} g_1g_1) + D_{\frac{1}{2}\frac{1}{2}}(J_y)(\tfrac{i}{2} g_2g_2 - \tfrac{i}{2} g_1g_1) + 0$$

$$= \tfrac{1}{4} g_1g_2 + \tfrac{1}{4} g_2g_1 + \tfrac{1}{4} g_2g_1 + \tfrac{1}{4} g_1g_2 + \tfrac{1}{4} g_1g_2$$

$$+ \tfrac{1}{4} g_2g_1 + \tfrac{1}{4} g_1g_2 + \tfrac{1}{4} g_2g_1$$

$$= g_1g_2 + g_2g_1 .$$

$$D_{\frac{1}{2}\frac{1}{2}}(J^2)g_2g_1 = (D_{\frac{1}{2}\frac{1}{2}}(J_x))^2 g_2g_1 + (D_{\frac{1}{2}\frac{1}{2}}(J_y))^2 g_2g_1 + (D_{\frac{1}{2}\frac{1}{2}}(J_z))^2 g_2g_1$$

$$= D_{\frac{1}{2}\frac{1}{2}}(J_x)(\tfrac{1}{2} g_1g_1 + \tfrac{1}{2} g_2g_2) + D_{\frac{1}{2}\frac{1}{2}}(J_y)(-\tfrac{i}{2} g_1g_1 + \tfrac{i}{2} g_2g_2) + 0$$

$$= \tfrac{1}{4} g_2g_1 + \tfrac{1}{4} g_2g_2 + \tfrac{1}{4} g_1g_2 + \tfrac{1}{4} g_2g_1 + \tfrac{1}{4} g_2g_1$$

$$+ \tfrac{1}{4} g_1g_2 + \tfrac{1}{4} g_1g_2 + \tfrac{1}{4} g_2g_1$$

$$= g_1g_2 + g_2g_i .$$

$$D_{\frac{1}{2}\frac{1}{2}}(J^2)g_2g_2 = (D_{\frac{1}{2}\frac{1}{2}}(J_x))^2 g_2g_2 + (D_{\frac{1}{2}\frac{1}{2}}(J_y))^2 g_2g_2 + (D_{\frac{1}{2}\frac{1}{2}}(J_z))^2 g_2g_2$$

$$= D_{\frac{1}{2}\frac{1}{2}}(J_x)(\tfrac{1}{2} g_1g_2 + \tfrac{1}{2} g_2g_1) + D_{\frac{1}{2}\frac{1}{2}}(J_y)(-\tfrac{i}{2} g_1g_2 - \tfrac{i}{2} g_2g_1)$$

$$+ D_{\frac{1}{2}\frac{1}{2}}(J_z)(-g_2g_2)$$

$$= \tfrac{1}{4} g_2g_2 + \tfrac{1}{4} g_1g_1 + \tfrac{1}{4} g_1g_1 + \tfrac{1}{4} g_2g_2 + \tfrac{1}{4} g_2g_2$$

$$- \tfrac{1}{4} g_1g_1 + \tfrac{1}{4} g_2g_2 - \tfrac{1}{4} g_1g_1 + g_2g_2$$

$$= 2g_2g_2 .$$

The tensors g_1g_1 and g_2g_2 are then eigentensors of

$$D_{\substack{11\\2\bar{2}}}(J_z)g_1g_1 = g_1g_1$$

and

$$D_{\substack{11\\2\bar{2}}}(J^2)g_1g_1 = 2g_1g_1$$

$$D_{\substack{11\\2\bar{2}}}(J_z)g_2g_2 = -g_2g_2$$

and

$$D_{\substack{11\\2\bar{2}}}(J^2)g_2g_2 = 2g_1g_1 \ .$$

Since these tensors are already normalized, $(\bar{g}_1\bar{g}_1):(g_1g_1) = 1$, $(\bar{g}_2\bar{g}_2):(g_2g_2) = 1$, they will serve as two of the vectors $\Phi_{\substack{11\\2\bar{2}}}(J,M)$. According to the above eigentensor equations, for g_1g_1 $M = 1$ and $J(J+1) = 2$ so $J = 1$. For g_2g_2, $M = -1$ and $J = 1$. Therefore

$$\Phi_{\substack{11\\2\bar{2}}}(1,1) = g_1g_1$$

$$\Phi_{\substack{11\\2\bar{2}}}(1,-1) = g_2g_2 \ .$$

The other basis tensors g_1g_2 and g_2g_1 are not eigentensors of $D_{\substack{11\\2\bar{2}}}(J^2)$ but certain linear combinations are:

$$D_{\substack{11\\2\bar{2}}}(J^2)(g_1g_2 + g_2g_1) = D_{\substack{11\\2\bar{2}}}(J^2)g_1g_2 + D_{\substack{11\\2\bar{2}}}(J^2)g_2g_1$$

$$= (g_1g_2 + g_2g_1) + (g_2g_1 + g_1g_2)$$

$$= 2(g_1g_2 + g_2g_1)$$

and $D_{\substack{11\\2\bar{2}}}(J^2)(g_1g_2 - g_2g_1) = D_{\substack{11\\2\bar{2}}}(J^2)g_1g_2 - D_{\substack{11\\2\bar{2}}}(J^2)g_2g_1$

$$= (g_1g_2 + g_2g_1) - (g_2g_1 + g_1g_2) = 0 .$$

Since both g_1g_2 and g_2g_1 are eigentensors of $D_{\substack{11\\2\bar{2}}}(J_z)$ with eigenvalue 0, so are these linear combinations. Although they have the same eigenvalue, these two tensors are still orthogonal

$$\overline{(g_1g_2 + g_2g_1)}\cdot(g_1g_2 - g_2g_1) = (\bar{g}_1\cdot g_1)(\bar{g}_2\cdot g_2) + (\bar{g}_2\cdot g_1)(\bar{g}_1\cdot g_2)$$

$$-(\bar{g}_2\cdot g_1)(\bar{g}_1\cdot g_2) - (\bar{g}_1\cdot g_1)(\bar{g}_2\cdot g_2)$$

$$= 1 - 1 = 0 .$$

They are not normalized since

$$(\bar{g}_1\bar{g}_2 + \bar{g}_2\bar{g}_1)\cdot(g_1g_2 + g_2g_1) = (\bar{g}_1\cdot g_1)(\bar{g}_2\cdot g_2) + (\bar{g}_1\cdot g_2)(\bar{g}_2\cdot g_1)$$

$$+(\bar{g}_2\cdot g_1)(\bar{g}_1\cdot g_2) + (\bar{g}_2\cdot g_2)(\bar{g}_1\cdot g_1)$$

$$= 1 + 0 + 0 + 1 = 2$$

and $(\bar{g}_1\bar{g}_2 - \bar{g}_2\bar{g}_1)\cdot(g_1g_2 - g_2g_1) = (\bar{g}_1\cdot g_1)(\bar{g}_2\cdot g_2) - (\bar{g}_1\cdot g_2)(\bar{g}_2\cdot g_1)$

$$-(\bar{g}_2\cdot g_1)(\bar{g}_1\cdot g_2) + (\bar{g}_2\cdot g_2)(\bar{g}_1\cdot g_1)$$

$$= 1 - 0 - 0 + 1 = 2 .$$

In each case the normalization factor is $2^{-1/2}$. In terms of the tensors $\Phi_{\substack{11\\2\bar{2}}}(J,M)$ this gives

$$\Phi_{\frac{1}{2}\frac{1}{2}}(1,0) = 2^{-1/2}(g_1 g_2 + g_2 g_1)$$

$$\Phi_{\frac{1}{2}\frac{1}{2}}(0,0) = 2^{-1/2}(g_1 g_2 - g_2 g_1) \ .$$

Combining all of these gives a complete orthonormal basis for $(U_2^{\otimes 1})_s \otimes (U_2^{\otimes 1})_s$ of simultaneous eigentensors of $D_{\frac{1}{2}\frac{1}{2}}(J_z)$ and $D_{\frac{1}{2}\frac{1}{2}}(J^2)$. Also, recalling that $g_1 = g(1,0) = \phi(\frac{1}{2},\frac{1}{2})$ and $g_2 = g(0,1) = \phi(\frac{1}{2},-\frac{1}{2})$, this may be written

$$\Phi_{\frac{1}{2}\frac{1}{2}}(1,1) = g_1 g_1 = g(1,0) \otimes g(1,0) = \phi(\tfrac{1}{2},\tfrac{1}{2}) \otimes \phi(\tfrac{1}{2},\tfrac{1}{2})$$

$$\Phi_{\frac{1}{2}\frac{1}{2}}(1,0) = 2^{-1/2}(g_1 g_2 + g_2 g_1) = 2^{-1/2}(g(1,0) \otimes g(0,1) + g(0,1) \otimes g(1,0))$$

$$= 2^{-1/2}(\phi(\tfrac{1}{2},\tfrac{1}{2}) \otimes \phi(\tfrac{1}{2},-\tfrac{1}{2}) + \phi(\tfrac{1}{2},-\tfrac{1}{2}) \otimes \phi(\tfrac{1}{2},\tfrac{1}{2}))$$

$$\Phi_{\frac{1}{2}\frac{1}{2}}(1,-1) = g_2 g_2 = g(0,1) \otimes g(0,1) = \phi(\tfrac{1}{2},-\tfrac{1}{2}) \otimes \phi(\tfrac{1}{2},-\tfrac{1}{2})$$

$$\Phi_{\frac{1}{2}\frac{1}{2}}(0,0) = 2^{-1/2}(g_1 g_2 - g_2 g_1) = 2^{-1/2}(g(1,0) \otimes g(0,1) - g(0,1) \otimes g(1,0))$$

$$= 2^{-1/2}(\phi(\tfrac{1}{2},\tfrac{1}{2}) \otimes \phi(\tfrac{1}{2},-\tfrac{1}{2}) - \phi(\tfrac{1}{2},\tfrac{1}{2}) \otimes \phi(\tfrac{1}{2},-\tfrac{1}{2}) \ .$$

The allowed values of angular momentum and the Clebsch-Gordon coefficients in the case $j_1 = j_2 = \frac{1}{2}$ may be read directly from these formulas. J may have the values 0 or 1. When J = 1, M may be -1, 0,

or +1. When J = 0, M must be 0. The Clebsch-Gordon coefficient $C_{\frac{1}{2}\frac{1}{2}}^{1}(m_1,m_2;J,M)$ is just the coefficient of $\phi(\frac{1}{2},m_1) \times \phi(\frac{1}{2},m_2)$ in the expansion of $\Phi_{\frac{1}{2}\frac{1}{2}}^{1}(J,M)$. $\phi(\frac{1}{2},m_1) \times \phi(\frac{1}{2},m_2)$ appears in the expansion only if $m_1 + m_2 = M$. Therefore

$$C_{\frac{1}{2}\frac{1}{2}}^{1}(m_1,m_2;J,M) = 0 \qquad \text{if } m_1 + m_2 \neq M .$$

In the other cases

$$C_{\frac{1}{2}\frac{1}{2}}^{1}(\tfrac{1}{2},\tfrac{1}{2};1,1) = 1$$

$$C_{\frac{1}{2}\frac{1}{2}}^{1}(\tfrac{1}{2},-\tfrac{1}{2};1,0) = 2^{-1/2}$$

$$C_{\frac{1}{2}\frac{1}{2}}^{1}(-\tfrac{1}{2},\tfrac{1}{2};1,0) = 2^{-1/2}$$

$$C_{\frac{1}{2}\frac{1}{2}}^{1}(-\tfrac{1}{2},-\tfrac{1}{2};1,-1) = 1$$

$$C_{\frac{1}{2}\frac{1}{2}}^{1}(\tfrac{1}{2},-\tfrac{1}{2};1,0) = 2^{-1/2}$$

$$C_{\frac{1}{2}\frac{1}{2}}^{1}(+\tfrac{1}{2},-\tfrac{1}{2};0,0) = 2^{-1/2}$$

$$C_{\frac{1}{2}\frac{1}{2}}^{1}(-\tfrac{1}{2},\tfrac{1}{2};0,0) = -2^{-1/2} .$$

5. The General $\Phi_{j_1 j_2}(J,M)$

The eigentensor $\Phi_{\frac{1}{2}\frac{1}{2}}^{1}(0,0)$ is of particular interest. The representation spaces are constructed from symmetric tensor but

$\Phi_{\substack{11\\22}}(0,0) = 2^{-1/2}(g_1g_2 - g_2g_1)$ is antisymmetric. Furthermore, not only is $\Phi_{\substack{11\\22}}(0,0)$ an eigentensor of both $D_j(J_z)$ and $D_j(J^2)$ with eigenvalue zero but also

$$D_{\substack{11\\22}}(J_x)(g_1g_2 - g_2g_1) = D_{\substack{11\\22}}(J_x)g_1g_2 - D_{\substack{11\\22}}(J_x)g_2g_1$$

$$= (\tfrac{1}{2} g_2g_2 + \tfrac{1}{2} g_1g_1) - (\tfrac{1}{2} g_2g_2 + \tfrac{1}{2} g_1g_1)$$

$$= 0$$

and $$D_{\substack{11\\22}}(J_y)(g_1g_2 - g_2g_1) = D_{\substack{11\\22}}(J_y)g_1g_2 - D_{\substack{11\\22}}(J_y)g_2g_1$$

$$= \tfrac{i}{2} g_2g_2 - \tfrac{i}{2} g_1g_1 - (-\tfrac{i}{2} g_1g_1 + \tfrac{i}{2} g_2g_2)$$

$$= 0.$$

$\Phi_{\substack{11\\22}}(0,0)$ is then an eigentensor of all the operators with eigenvalue zero.

Now consider the product of a tensor of the type $\Phi_{j_1j_2}(J,M)$ with one of the type $\phi(j,m)$. The product is not necessarily in any of the tensor spaces defined above as it may not have the proper symmetry properties but the operators may be extended to $\Phi_{j_1j_2}(J,M) \times \phi(j,m)$ by

$$D(J_x)(\Phi_{j_1j_2}(J,M) \times \phi(j,m)) = D_{j_1j_2}(J_x)\,\Phi_{j_1j_2}(J,M) \times \phi(j,m)$$

$$+\Phi_{j_1j_2}(J,M) \times D_j(J_x)\,\phi(j,m)$$

and similarly for the other operators.

For the $D(J_z)$ operator one has

$$D_j(J_z)(\Phi_{j_1 j}(J,M) \,\square\, \phi(j,m)) = D_{j_1 j_2}(J_x)\, \Phi_{j_1 j_2}(J,M) \,\square\, \phi(j,m)$$

$$+\Phi_{j_1 j_2}(J,M) \,\square\, D_j(J_x)\, \phi(j,m)$$

$$= (M + m)\, \Phi_{j_1 j_2}(J,M) \ .$$

For $D(J^2)$ defined as usual by

$$D(J^2) = (D(J_x))^2 + (D(J_y))^2 + (D(J_z))^2 \ ,$$

the tensor $\Phi_{j_1 j_2}(J,M)$ is not an eigentensor. This is because, again, there are cross terms of the form $D_{j_1 j_2}(J_x)\, \Phi_{j_1 j_2}(J,M) \,\square\, D_j(J_x)\, \phi(j,m)$. Since, in general, the tensors are not eigentensors of $D_j(J_x)$ the cross terms give rise to more complicated tensors. The special case, $\Phi_{j_1 j_2}(J,M) = \Phi_{\frac{1}{2}\frac{1}{2}}(0,0)$, however, gives particularly useful results. In this case,

$$D(J_z)(\Phi_{\frac{1}{2}\frac{1}{2}}(0,0) \,\square\, \phi(j,m)) = \Phi_{\frac{1}{2}\frac{1}{2}}(0,0) \,\square\, D_j(J_z)\, \phi(j,m)$$

$$= m\, \Phi_{\frac{1}{2}\frac{1}{2}}(0,0) \,\square\, \phi(j,m)$$

$$D(J^2)(\Phi_{\frac{1}{2}\frac{1}{2}}(0,0) \,\square\, \phi(j,m)) = (D(J_x))^2\, (\Phi_{\frac{1}{2}\frac{1}{2}}(0,0) \,\square\, \phi(j,m)$$

$$+(D(J_y))^2\, (\Phi_{\frac{1}{2}\frac{1}{2}}(0,0) \,\square\, \phi(j,m)$$

$$+(D(J_z))^2\, (\Phi_{\frac{1}{2}\frac{1}{2}}(0,0) \,\square\, \phi(j,m)$$

$$= \Phi_{\frac{1}{2}\frac{1}{2}}(0,0) \,\square\, D_j(J^2)\, \phi(j,m)$$

$$= j(j+1)\, \Phi_{\frac{1}{2}\frac{1}{2}}(0,0) \,\square\, \phi(j,m)$$

since all operators on $\Phi_{\frac{11}{22}}(0,0)$ yield zero. In the same way the operators can be extended to tensor products of the type $(\Phi_{\frac{11}{22}}(0,0))^{\alpha n} \alpha \phi(j,m)$, the tensor product of n copies of $\Phi_{\frac{11}{22}}(0,0)$ with $\phi(j,m)$. For these as well,

$$D(J_z)((\Phi_{\frac{11}{22}}(0,0))^{\alpha n} \alpha \phi(j,m)) = m(\Phi_{\frac{11}{22}}(0,0))^{\alpha n} \alpha \phi(j,m)$$

and $$D(J^2)((\Phi_{\frac{11}{22}}(0,0))^{\alpha n} \alpha \phi(j,m)) = j(j+1)(\Phi_{\frac{11}{22}}(0,0))^{\alpha n} \alpha \phi(j,m).$$

In this way a tensor can be constructed with the same eigenvalue as $\phi(j,m)$ but of higher order. This tensor product is not usually a member of $(U_2^{\alpha 2j_1})_s \alpha (U_2^{\alpha 2j_2})_s$ since it may lack the necessary symmetry properties. However, since the operators $D(J_x)$, etc., depend only on the number of g_1's and g_2's, they commute with permutation operators so that the tensor may be symmetrized without changing the fact that it is an eigentensor.

Let $\phi(J,M)$ be a tensor in $(U_2^{\alpha 2J})_s$ which is normalized and is an eigentensor of $D_J(J_z)$ with eigenvalue M and of $D_J(J^2)$ with eigenvalue J(J+1). Then the tensor product $(\Phi_{\frac{11}{22}}(0,0))^{\alpha(j_1+j_2-J)} \alpha \phi(J,M)$ has the same eigenvalues M and J(J+1) for arbitrary j_1 and j_2 with J less than or equal to $j_1 + j_2$. Let Sym (j_1,j_2) be the operator which symmetrizes the first $2j_1$ and last $2j_2$ places in the tensor. For example, Sym $(1,\frac{1}{2})g_1g_2g_2 = g_1g_2g_2 + g_2g_1g_2$ and Sym $(1,1)g_1g_2g_1g_2 = g_1g_2g_1g_2 + g_2g_1g_1g_2 + g_1g_2g_2g_1 + g_2g_1g_2g_1$.

Since $\phi(J,M)$ is a tensor of order $2J$ and $(\Phi_{\frac{1}{2}\frac{1}{2}}(0,0))^{\alpha(j_1+j_2-J)}$ is a tensor of order $2(j_1+j_2-J)$, their tensor product, $(\Phi_{\frac{1}{2}\frac{1}{2}}(0,0))^{\alpha(j_1+j_2-J)} \alpha \phi(J,M)$, is a tensor of order $2J + 2(j_1+j_2-J) = 2j_1 + 2j_2$. Now, one first applies the operator P_R which changes the order of the individual terms so that the odd terms of $(\Phi_{\frac{1}{2}\frac{1}{2}}(0,0))^{\alpha(j_1+j_2-J)}$ precede the terms of $\phi(J,M)$ while the even terms follow. This will simplify the symmetrization. After the first $2j_1$ and last $2j_2$ places have been symmetrized, the tensor is contained in $(U_2^{\alpha 2j_1})_s \alpha (U_2^{\alpha 2j_2})_s$. Furthermore, it still has eigenvalues M and J(J+1). If N is the normalization constant, $\Phi_{j_1 j_2}(J,M) = N \text{ sym } (j_1,j_2) P_R ((\Phi_{\frac{1}{2}\frac{1}{2}}(0,0))^{\alpha(j_1+j_2-J)} \alpha \phi(J,M))$. This gives a method of constructing the eigentensors $\Phi_{j_1 j_2}(J,M)$ in $(U_2^{\alpha 2j_1})_s \alpha (U_2^{\alpha 2j_2})_s$ from the eigentensors $\phi(J,M)$ in $(U_2^{\alpha 2J})_s$ which are already known.

To begin with, since the tensor must be normalized after the symmetrization anyway, it will be simpler to ignore the normalization factors from the beginning. That is, instead of $\Phi_{\frac{1}{2}\frac{1}{2}}(0,0)$, $g_1g_2 - g_2g_1$ will be used. Instead of $\phi(J,M)$, $g(J+M,J-M)$ will be used. Now the symmetrization proceeds as follows. Each monomial term with a g_1's and b g_2's in the first $2j_1$ places, c g_1's and d g_2's in the last $2j_2$ places is expanded by the symmetrization operator into the symmetrical form $g(a,b) \alpha g(c,d)$. The integers a, b, c, and d are not arbitrary since $a+b = 2j_1$, $c+d = 2j_2$. In addition, since there are exactly $j_1 + j_2 - J$ g_1's and $j_1 + j_2 - J$ g_2's in each monomial term of $(g_ig_2 - g_2g_1)^{\alpha(j_1+j_2-J)}$ and exactly $J + M$ g_1's and $J - M$ g_2's

in each monomial term of $g(J+M,J-M)$,

$$a+c = (j_1+j_2-J) + J + M = j_1 + j_2 + M$$

and $$b+d = (j_1+j_2-J) + J - M = j_1 + j_2 - M .$$

If m is defined by $a = j_1 + m$, then $b = 2j_1 - (j_1 + m) = j_1 - m$, $c = j_1 + j_2 + M - (j_1 + m) = j_2 + M - m$, and $d = 2j_2 - (j_2 + M - m) = j_2 - M + m$. Thus, $(g_1g_2 - g_2g_1)^{\boxtimes(j_1+j_2-J)} \boxtimes \phi(J,M)$ will be expanded into the sum $\sum_m a_m(j_1 + m, j_1 - m) \boxtimes g(j_2 + M - m, j_2 - M + m)$ where the factor a_m is the total number of positive terms which expand to $g(j_1 + m, j_1 - m) \boxtimes g(j_2 + M - m, j_2 - M + m)$ minus the total number of negative terms which expand to $g(j_1 + m, j_1 - m)$.

Let b equal the number of g_2's in the first $j_1 + j_2 - J$ places. Since these are selected from the $j_1 + j_2 - J$ g_1's and g_2's in the odd places of the monomial terms of $(g_1g_2 - g_2g_1)^{\boxtimes(j_1+j_2-J)}$, there are $\binom{j_1+j_2-J}{b}$ different ways such a term will appear. To have $j_1 - m$ g_2's in the first $2j_1$ places there must be $j_1 - m - b$ g_2's in the next $2j_1 - (j_1 + j_2 - J) = j_1 - j_2 + J$ places. Since these come from the $(J-M)$ g_2's of the terms in $g(J+M,J-M)$, there are $\binom{J+j_1-j_2}{j_1-m-b}$ ways this can occur.

Finally, the form can be fixed by requiring that there be $(J-M) - (j_1 - m - b) = J - j_1 - (M-m) + b$ g_2's in the next $2J - (J + j_1 - j_2) = J - j_1 + j_2$ places. There are $\binom{J+j_2-j_1}{J-j_1-(M-m)+b}$ ways this can occur. Therefore, there will be

$$\sum_{b=0}^{j_1-m} \begin{pmatrix} j_1+j_2-J \\ b \end{pmatrix} \begin{pmatrix} J+j_1-j_2 \\ j_1-m-b \end{pmatrix} \begin{pmatrix} J+j_2-j_1 \\ J-j_1-(M-m)+b \end{pmatrix}$$

terms in the original tensor product which give rise to $g(j_1+m, j_1-m) \propto g(j_2+M-m, j_2-M+m)$. Since those terms with an odd number of g_2's in the first $j_1 + j_2 - J$ places one has

$$a_m = \sum_{b=0}^{j_1-m} (-1)^b \begin{pmatrix} j_1+j_2-J \\ b \end{pmatrix} \begin{pmatrix} J+j_1-j_2 \\ j_1-m-b \end{pmatrix} \begin{pmatrix} J+j_2-j_1 \\ J-j_1-(M-m)+b \end{pmatrix} .$$

This yields

$$\Phi_{j_1 j_2}(J,M) = N_o \sum_m \sum_b (-1)^b \begin{pmatrix} j_1+j_2-J \\ b \end{pmatrix} \begin{pmatrix} J+j_1-j_2 \\ j_1-m-b \end{pmatrix} \begin{pmatrix} J+j_2-j_1 \\ J-j_1-(M-m)+b \end{pmatrix}$$

$$g(j_1+m, j_1-m) \propto g(j_2+M-m, j_2-M+m)$$

where the summation extends over all values of m and b such that the numbers in the binomial coefficients are non-negative integers and N_o is the normalizing factor

$$N_o = \left(\sum_m \left(\sum_b (-1)^b \begin{pmatrix} j_1+j_2-J \\ b \end{pmatrix} \begin{pmatrix} J+j_1-j_2 \\ j_1-m-b \end{pmatrix} \begin{pmatrix} J+j_2-j_1 \\ J-j_1-(M-m)+b \end{pmatrix} \right)^2 \right.$$

$$\left. \frac{(j_1+m)!(j_1-m)!(j_2+M-m)!(j_2-M+m)!}{(2j_1)!(2j_2)!} \right)^{-1/2} .$$

Then

$$\Phi_{j_1 j_2}(J,M) = N \sum_m \sum_b (-1)^b \begin{pmatrix} j_1+j_2-J \\ b \end{pmatrix} \begin{pmatrix} J+j_1-j_2 \\ j_1-m-b \end{pmatrix} \begin{pmatrix} J+j_2-j_1 \\ J-j_1-(M-m)+b \end{pmatrix}$$

$$((j_1+m)!(j_1-m)!(j_2+M-m)!(j_2-M+m)!)^{1/2} \propto \phi(j_1,m) \propto \phi(j_2,m)$$

where $N = N_o((2j_1)!(2j_2)!)^{-1/2}$.

Since the Clebsch-Gordon coefficients $C_{j_1 j_2}(m_1,m_2;J,M)$ are just the coefficients in this expansion they may now be written down easily. First, since only tensors of the form $\phi(j,m) \propto \phi(j_2,M-m)$ appear in the expansion, $C_{j_1 j_2}(m_1,m_2;J,M) = 0$ if $m_1 + m_2 \neq M$. When $m_1 + m_2 = M$,

$$C_{j_1 j_2}(m_1,m_2;J,M) = N \sum_b (-1)^b \begin{pmatrix} j_1+j_2-J \\ b \end{pmatrix} \begin{pmatrix} J+j_1-j_2 \\ j_1-m_1-b \end{pmatrix} \begin{pmatrix} J+j_2-j_1 \\ J-j_1-m_2+b \end{pmatrix}$$

$$((j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!)^{1/2}$$

where N has the same value as before.

6. Allowed Values of J and M

The numbers in the binomial coefficients must be non-negative integers. That is,

$$j_1 + j_2 - J \geq 0 \quad \text{or} \quad j_1 + j_2 \geq J$$
$$J + j_2 - j_1 \geq 0 \quad \text{or} \quad J \geq j_1 - j_2$$
$$J + j_1 - j_2 \geq 0 \quad \text{or} \quad J \geq j_2 - j_1 .$$

Therefore, J may take on any values at integral steps between $|j_1 - j_2|$ and $j_1 + j_2$. As noted before, M may take on values at integral steps between -J and J and $m_1 + m_2 = M$.

7. Special Cases

If $J = j_1 + j_2$, the first binomial coefficient is nonzero only if $b = 0$. Then the sum collapses to

$$C_{j_1 j_2}(m_1,m_2;j_1+j_2,m_1+m_2) = N \begin{pmatrix} J+j_1-j_2 \\ j_1-m_1 \end{pmatrix} \begin{pmatrix} J+j_2-j_1 \\ j_2-m_2 \end{pmatrix}$$

$$((j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!)^{1/2}$$

$$= N \frac{(J+j_1-j_2)!(J+j_2-j_1)!}{(J-j_2+m_1)!(j_2-m_2)!} \left[\frac{(j_1+m_1)!(j_2+m_2)!}{(j_1-m_1)!(j_2-m_2)!}\right]^{1/2}$$

$$= N \frac{(2j_1)!(2j_2)!}{(j_1+m_1)!(j_2+m_2)!} \left[\frac{(j_1+m_1)!(j_2+m_2)!}{(j_1-m_1)!(j_2-m_2)!}\right]^{1/2}.$$

The normalizing factor N is now given by

$$N = \left[\sum_m \begin{pmatrix} j_1+j_1-j_2 \\ j_1-m \end{pmatrix}^2 \begin{pmatrix} J+j_2-j_1 \\ J-j_1-(M-m) \end{pmatrix}^2 \frac{(j_1+m)!(j_1-m)!(j_2+M-m)!(j_2-M+m)!}{(2j_1)!(2j_2)!}\right]^{-1/2}$$

$$= \left[\sum_m \left[\frac{(2j_1)!(2j_2)!}{(j_1+m)!(j_1-m)!(j_2+(M-m))!(j_2-(M-m))!}\right]^2 \right.$$

$$\left. \frac{(j_1+m)!(j_1-m)!(j_1+M-m)!(j_2-M+m)!}{(2j_1)!(2j_2)!}\right]^{-1/2}$$

$$= \left[\sum_m \begin{pmatrix} 2j_1 \\ j_1+m \end{pmatrix} \begin{pmatrix} 2j_2 \\ j_2+(M-m) \end{pmatrix}\right]^{-1/2}$$

$$= \begin{pmatrix} 2J \\ J+M \end{pmatrix}^{-1/2} = \left[\frac{(J+M)!(J-M)!}{(2J)!}\right]^{1/2}$$

so

$$C_{j_1 j_2}(m_1,m_2;j_1+j_2,m_1+m_2) = \left[\frac{(j_1+j_2+m_1+m_2)!(j_1+j_2-m_1-m_2)!}{(2j_1+2j_2)!}\right]^{1/2}$$

$$\left[\frac{(2j_1)!(2j_2)!}{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}\right]^{1/2}.$$

If $J = j_2 - j_1$, the second binomial coefficient is nonzero only if $b = j_1 - m_1$ so the sum reduces to

$$C_{j_1 j_2}(m_1,m_2;j_2-j_1,m_1+m_2) = N\,(-1)^{j_1-m_1}\begin{pmatrix}2j_1\\ j_1-m_1\end{pmatrix}\begin{pmatrix}2J\\ J-M\end{pmatrix}$$

$$((j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!)^{1/2}.$$

N is given by

$$N = \left\{\sum_m (-1)^{2j_1-2m}\begin{pmatrix}2j_1\\ j_1-m\end{pmatrix}^2\begin{pmatrix}2J\\ J-M\end{pmatrix}^2 (j_1+m)!(j_1-m)!(j_2+M-m)!(j_2-M+m)!\right\}^{-1/2}.$$

After reducing this in the same way as before,

$$C_{j_1 j_2}(m_1,m_2,j_2-j_1,m_1+m_2) = \left\{\frac{(2j_1)!(2j_2)!(j_2+m_2)!(j_2-m_2)!}{(2j_1+2j_2)!(J+M)!(J-M)!(j_1+m_1)!(j_1-m_1)!}\right\}^{1/2}$$

$$(-1)^{j_1-m_1}.$$

If $j_1 = \frac{1}{2}$, these are the only two possibilities, $J = j_2 + \frac{1}{2}$ or $J = j_2 - \frac{1}{2}$. In the case $J = j_2 + \frac{1}{2}$, $m_1 = \frac{1}{2}$,

$$C_{\frac{1}{2}j_2}(\tfrac{1}{2},m_2;j_2+\tfrac{1}{2},m_2+\tfrac{1}{2}) = \left\{\frac{(j_2+m_2+1)!(j_2-m_2)!}{(2j_2+1)!}\,\frac{(2j_2)!}{(j_2+m_2)!(j_2-m_2)!}\right\}^{1/2}$$

$$= \left\{\frac{j_2+m_2+1}{2j_2+1}\right\}^{1/2}.$$

If $J = j_2 + \frac{1}{2}$, $m_1 = -\frac{1}{2}$,

$$C_{\frac{1}{2}j_2}\left(\tfrac{1}{2},m_2;j_2+\tfrac{1}{2},m_2-\tfrac{1}{2}\right) = \left[\frac{(j_2+m_2)!(j_2-m_2+1)!(2j_1)!}{(2j_2+1)!(j_2+m_2)!(j_2-m_2)!}\right]^{1/2} = \left[\frac{j_2-m_2+1}{2j_2+1}\right]^{1/2}.$$

If $J = j_2 - \frac{1}{2}$, $m_1 = \frac{1}{2}$

$$C_{\frac{1}{2}j_2}\left(\tfrac{1}{2},m_2;j_2-\tfrac{1}{2},m_2+\tfrac{1}{2}\right) = \left[\frac{(2j_2)!(j_2+m_2)!(j_2-m_2)!}{(2j_2+1)!(j_2+m_2)!(j_2-m_2-1)!}\right]^{1/2}\left[\frac{j_2-m_2}{2j_2+1}\right]^{1/2}.$$

If $J = j_2 - \frac{1}{2}$, $m_1 = -\frac{1}{2}$,

$$C_{\frac{1}{2}j_2}\left(-\tfrac{1}{2},m_2;j_2-\tfrac{1}{2},m_2-\tfrac{1}{2}\right) = -\left[\frac{(2j_2)!(j_2+m_2)!(j_2-m_2)!}{(2j_2+1)!(j_2+m_2-1)!(j_2-m_2)!}\right]^{1/2} = -\left[\frac{j_2+m_2}{2j_2+1}\right]^{1/2}.$$

8. Conclusion

This tensor method of representation appears to have several advantages over the standard methods of matrix representations. The first is a matter of elegance. The Clebsch-Gordon coefficients are defined abstractly in terms of the representations but using the matrix representations the Clebsch-Gordon coefficients are calculated by writing the matrix elements in terms of the original Euler angles of the rotations. In the tensor representation method, the Clebsch-Gordon coefficients are computed directly from the representation without reference to the physical interpretation of the group.

This leads to a more important point: The question of generalizations. The rotations operators are not the only operators which lead to SU(2) or similar groups. A set of operators on the nucleons, called the isospin operators, has group properties identical with the rotation group and also leads to the group SU(2). The

Clebsch-Gordon coefficients can then be used to describe the way the nucleons combine. Since the group is still SU(2), exactly the same Clebsch-Gordon coefficients may be used but the same general idea has led to other groups for which this is not true. Extension of the isospin operators to include unitary spin leads to the group SU(3). This leads to a classification procedure embracing other elementary particles. Other natural extensions have led to SU(4) [12], SU(6), and SU(9) [8]. Some researches appear to indicate that to completely understand even simple cases consideration of the general SU(n) may be necessary [2].

While all of these groups are related to SU(2), they do not have as simple a physical interpretation as the rotation group. The standard method of calculating Clebsch-Gordon coefficients, due to the dependence on the Euler angles, cannot be extended to groups other than SU(2). The calculations of Clebsch-Gordon coefficients which have been made are very long and difficult. Often papers are written on the calculation of one of the coefficients [7].

The tensor representation, however, has a natural generalization to SU(n). The operators are considered operators on a space of tensors on U_n, the unitary space of dimension n. A tensor method similar to the one given here has been used to calculate possible total eigenvalues in SU(3) although no attempt was made to calculate the Clebsch-Gordon coefficients [3].

BIBLIOGRAPHY

1. Boerner, Hermann: Representations of Groups, North-Holland Publishing Co., Amsterdam, 1964.

2. Capps, Richard H.: "Saturation of SU(3)-Symmetric Hadron Forces," Phys. Rev. (2) 144 (1966), 1293-1299.

3. Coleman, Sidney: "The Clebsch-Gordon Series for SU(3)," J. Math. Phys. 5 (1964), 1343-1344.

4. Halmos, Paul R.: Finite Dimensional Vector Spaces, D. Van Nostrand Co. Inc., Princeton, 1958.

5. Jacobson, Nathon: Lie Algebras, Interscience Publishers, New York, 1962.

6. Lang, Serge: Algebra, Addison-Wesley Publishing Co., Reading, Mass., 1965.

7. Schülke, Lothar: "The SU(6) Clebsch-Gordon Coefficients for the Reduction of (35)⊠(35) to (35) and (56)⊠(35) to (56)," Z. Physik 183 (1965), 424-427.

8. Singh, Vienda: "SU(a) Symmetry of Bargons and Mesons," Phys. Rev. Letters 15 (1965), 271-274.

9. Tinkhom, Michael: Group Theory and Quantum Mechanics, McGraw-Hill Book Company, New York, 1964.

10. Weyl, Hermann: The Theory of Groups and Quantum Mechanics, Dover Publications, Inc., 1955.

11. Wigner, Eugene P.: Group Theory and its Application to the Quantum Mechanics of Atomic Spectra, Academic Press, New York, 1959.

12. Woo Ching-hung and Dragt, Alex J.: "Some Simple Consequences of SU(4) Symmetry," Phys. Rev. (2) 139 (1965), B945-B946.

BIOGRAPHICAL SKETCH

George Edward Ivey was born on May 9, 1944, at Montezuma, Georgia. In June, 1962, he graduated from Centennial High School, Gresham, Oregon. In June, 1966, he received the degree of Bachelor of Science with a major in mathematics from Massachusetts Institute of Technology. In 1966, he entered graduate school at the University of Florida. From that time until the present, he has pursued his work toward the degree of Doctor of Philosophy.

George Ivey is married to the former Sandra Jean Benjamin.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

James K. Brooks, Chairman
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Charles P. Luehr, Co-Chairman
Assistant Professor of Mathematics

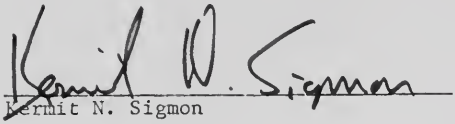
I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Zoran R. Pop-Stojanovic
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

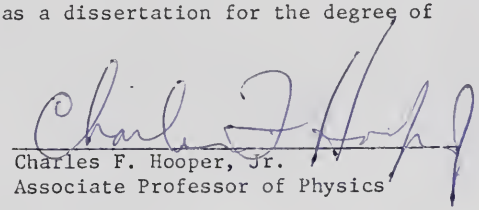
Arun K. Varma
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



Kermit N. Sigmon
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



Charles F. Hooper, Jr.
Associate Professor of Physics

This dissertation was submitted to the Department of Mathematics in the College of Arts and Sciences and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

June, 1972

Dean, Graduate School

SM27 14.1.17.